On Schemes for Exponential Decay

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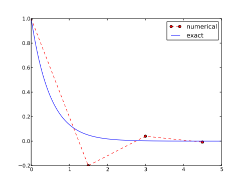

Goal

The primary goal of this demo talk is to demonstrate how to write talks with DocOnce and get them rendered in numerous HTML formats.

Layout

This version utilizes beamer slides with the theme `vintage`.

Problem setting and methods

Results

Problem setting and methods

We aim to solve the (almost) simplest possible differential equation problem

$$u'(t) = -au(t) \qquad (1)$$

$$u(0) = I \qquad (2)$$

Here,

- $t \in (0, T]$
- a, I, and T are prescribed parameters
- $u(t)$ is the unknown function
- The ODE (1) has the initial condition (2)

The ODE problem is solved by a finite difference scheme

- Mesh in time: $0 = t_0 < t_1 \cdots < t_N = T$
- Assume constant $\Delta t = t_n - t_{n-1}$
- u^n: numerical approx to the exact solution at t_n

The θ rule,

$$u^{n+1} = \frac{1 - (1-\theta)a\Delta t}{1 + \theta a\Delta t} u^n, \quad n = 0, 1, \ldots, N-1$$

contains the Forward Euler ($\theta = 0$), the Backward Euler ($\theta = 1$), and the Crank-Nicolson ($\theta = 0.5$) schemes.

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The Forward Euler scheme explained

http://youtube.com/PtJrPEIHNJw

Implementation

Implementation in a Python function:

```python
def solver(I, a, T, dt, theta):
    """Solve u'=-a*u, u(0)=I, for t in (0,T]; step: dt."""
    dt = float(dt)                # avoid integer division
    N = int(round(old_div(T,dt)))      # no of time intervals
    T = N*dt                      # adjust T to fit time step dt
    u = zeros(N+1)                # array of u[n] values
    t = linspace(0, T, N+1)  # time mesh

    u[0] = I                      # assign initial condition
    for n in range(0, N):         # n=0,1,...,N-1
        u[n+1] = (1 - (1-theta)*a*dt)/(1 + theta*dt*a)*u[n]
    return u, t
```

How to use the solver function

A complete main program

```
# Set problem parameters
I = 1.2
a = 0.2
T = 8
dt = 0.25
theta = 0.5
|\pause|

from solver import solver, exact_solution
u, t = solver(I, a, T, dt, theta)
|\pause|

import matplotlib.pyplot as plt
plt.plot(t, u, t, exact_solution)
plt.legend(['numerical', 'exact'])
plt.show()
```

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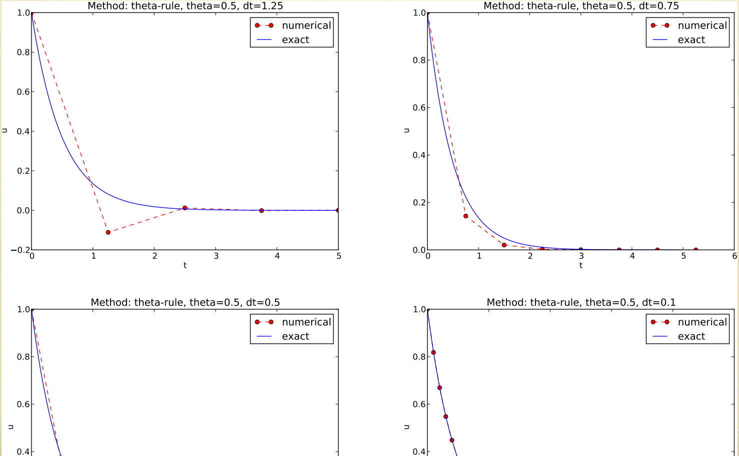
The Crank-Nicolson method shows oscillatory behavior for not sufficiently small time steps, while the solution should be monotone



The artifacts can be explained by some theory

Exact solution of the scheme:

$$u^n = A^n, \quad A = \frac{1-(1-\theta)a\Delta t}{1+\theta a\Delta t}\,.$$

Key results:

- Stability: $|A| < 1$
- No oscillations: $A > 0$
- $\Delta t < 1/a$ for Forward Euler ($\theta = 0$)
- $\Delta t < 2/a$ for Crank-Nicolson ($\theta = 1/2$)

Concluding remarks:
Only the Backward Euler scheme is guaranteed to always give qualitatively correct results.

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